

Your model may not always fit the experimental situation perfectly, but you should try to choose a model that *best fits* the population relative frequency histogram. The better the model approximates reality, the better your inferences will be. Fortunately, many continuous random variables have mound-shaped frequency distributions, such as the data in Figure 6.1(d). The **normal probability distribution** provides a good model for describing this type of data.

THE NORMAL PROBABILITY DISTRIBUTION

6.2

Continuous probability distributions can assume a variety of shapes. However, a large number of random variables observed in nature possess a frequency distribution that is approximately mound-shaped or, as the statistician would say, is approximately a normal probability distribution. The formula that generates this distribution is shown next.

NORMAL PROBABILITY DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad -\infty < x < \infty$$

The symbols e and π are mathematical constants given approximately by 2.7183 and 3.1416, respectively; μ and σ ($\sigma > 0$) are parameters that represent the population mean and standard deviation, respectively.

The graph of a normal probability distribution with mean μ and standard deviation σ is shown in Figure 6.5. The mean μ locates the *center* of the distribution, and the distribution is *symmetric* about its mean μ . Since the total area under the normal probability distribution is equal to 1, the symmetry implies that the area to the right of μ is .5 and the area to the left of μ is also .5. The *shape* of the distribution is determined by σ , the population standard deviation. As you can see in Figure 6.6, large values of σ reduce the height of the curve and increase the spread; small values of σ increase the height of the curve and reduce the spread. Figure 6.6 shows three normal probability distributions with different means and standard deviations. Notice the differences in shape and location.

FIGURE 6.5
Normal probability
distribution

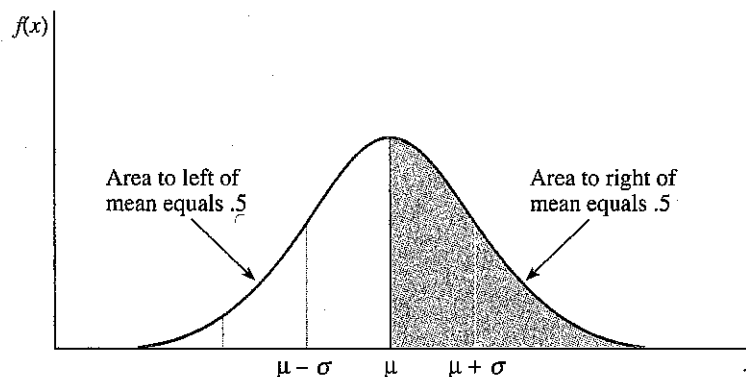
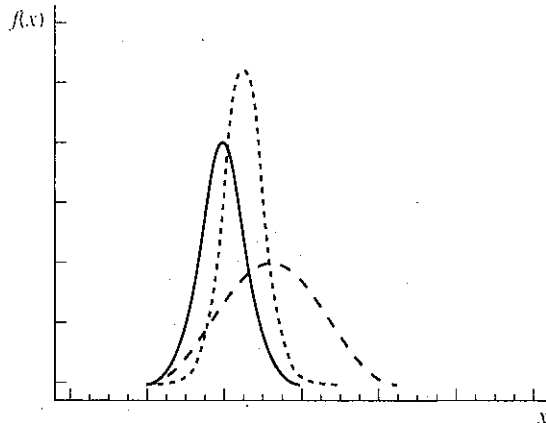


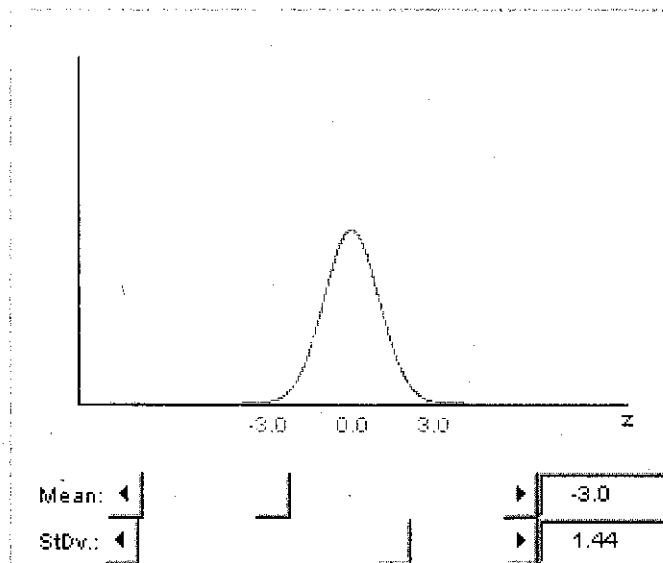
FIGURE 6.6
Normal probability
distributions with differing
values of μ and σ



APPLET

The Java applet called **Visualizing Normal Curves** gives a visual display of the normal distribution for values of μ between -10 and $+8$ and for values of σ between $.5$ and 1.8 . The dark blue curve is the standard normal z with mean 0 and standard deviation 1 . You can use this applet to compare its shape to the shape of other normal curves (the red curve on your monitor, light blue in Figure 6.7) by moving the sliders to change the mean and standard deviation. What happens when you change the mean? When you change the standard deviation?

FIGURE 6.7
Visualizing Normal
Curves applet



You rarely find a variable with values that are infinitely small ($-\infty$) or infinitely large ($+\infty$). Even so, many *positive* random variables (such as heights, weights, and times) have distributions that are well approximated by a normal distribution. According to the Empirical Rule, almost all values of a normal random variable lie in the interval $\mu \pm 3\sigma$. As long as the values within three standard deviations of the mean are *positive*, the normal distribution provides a good model to describe the data.

TABULATED AREAS OF THE NORMAL PROBABILITY DISTRIBUTION

6.3

To find the probability that a normal random variable x lies in the interval from a to b , we need to find the area under the normal curve between the points a and b (see Figure 6.2). However (see Figure 6.6), there are an infinitely large number of normal distributions—one for each different mean and standard deviation. A separate table of areas for each of these curves is obviously impractical. Instead, we use a standardization procedure that allows us to use the same table for all normal distributions.

The Standard Normal Random Variable

A normal random variable x is **standardized** by expressing its value as the number of standard deviations (σ) it lies to the left or right of its mean μ . This is really just a change in the units of measure that we use, as if we were measuring in inches rather than in feet! The standardized normal random variable, z , is defined as

$$z = \frac{x - \mu}{\sigma}$$

or equivalently,

$$x = \mu + z\sigma$$

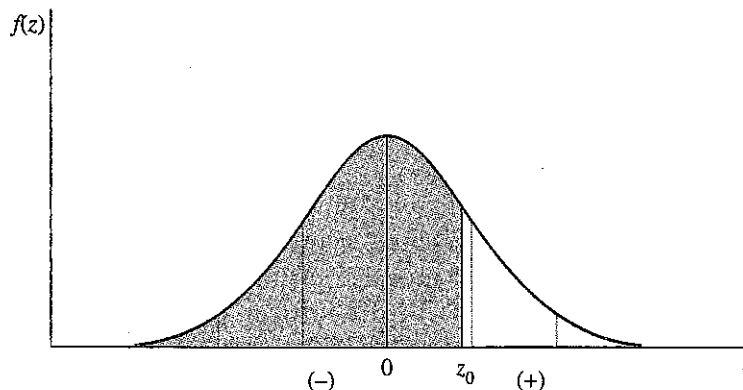
From the formula for z , we can draw these conclusions:

- When x is less than the mean μ , the value of z is negative.
- When x is greater than the mean μ , the value of z is positive.
- When $x = \mu$, the value of $z = 0$.

The probability distribution for z , shown in Figure 6.8, is called the **standardized normal distribution** because its mean is 0 and its standard deviation is 1. Values of z on the left side of the curve are negative, while values on the right side are positive. The area under the standard normal curve to the left of a specified value of z —say, z_0 —is the probability $P(z \leq z_0)$. This **cumulative area** is recorded in Table 3 of Appendix I and is shown as the shaded area in Figure 6.8. An abbreviated version of Table 3 is given in Table 6.1. Notice that the table contains both positive and negative values of z . The left-hand column of the table gives the value of z correct to the tenth place; the second decimal place for z , corresponding to hundredths, is given across the top row.

MY TIP
Area under the z -curve equals 1.

FIGURE 6.8
Standardized normal distribution



Abbreviated Version of Table 3 in Appendix I
Table 3. Areas Under the Normal Curve

TABLE 6.1

<i>z</i>	.00	.01	.02	.0309
-3.4	.0003	.0003	.0003	.0003		
-3.3	.0005	.0005	.0005	.0004		
-3.2	.0007	.0007	.0006	.0006		
-3.1	.0010	.0009	.0009	.0009		
-3.0	.0013	.0013	.0013	.00120010
-2.9	.0019	.	.	.		
-2.8	.0026	.	.	.		
-2.7	.0035	.	.	.		
-2.6	.0047					
-2.5	.0062					
.	.					
.	.					
-2.0	.0228					
.	.					
.	.					

Table 3. Areas Under the Normal Curve (continued)

<i>z</i>	.00	.01	.02	.03	.0409
0.0	.5000	.5040	.5080	.5120	.5160		
0.1	.5398	.5438	.5478	.5517	.5557		
0.2	.5793	.5832	.5871	.5910	.5948		
0.3	.6179	.6217	.6255	.6293	.6331		
0.4	.6554	.6591	.6628	.6664	.67006879
0.5	.6915	.	.	.			
0.6	.7257	.	.	.			
0.7	.7580	.	.	.			
0.8	.7881						
0.9	.8159						
.	.						
.	.						
.	.						
2.0	.9772						

EXAMPLE

6.3

Find $P(z \leq 1.63)$. This probability corresponds to the area to the left of a point $z = 1.63$ standard deviations to the right of the mean (see Figure 6.9).

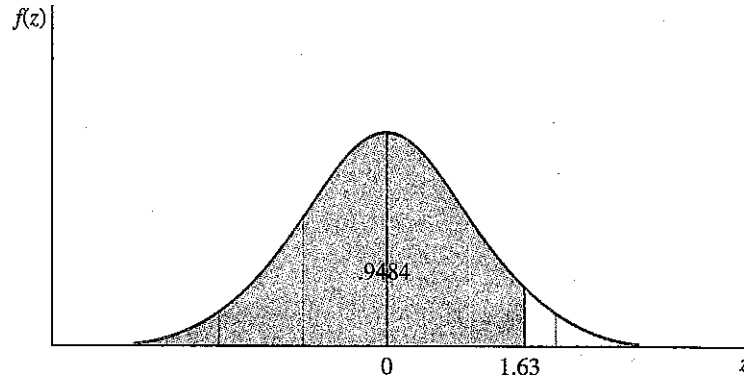
**TIP**

$$P(z \leq 1.63) = P(z < 1.63)$$

Solution The area is shaded in Figure 6.9. Since Table 3 in Appendix I gives areas under the normal curve to the left of a specified value of z , you simply need to find the tabled value for $z = 1.63$. Proceed down the left-hand column of the table to $z = 1.6$ and across the top of the table to the column marked .03. The intersection of this row and column combination gives the area .9484, which is $P(z \leq 1.63)$.

FIGURE 6.9

Area under the standard normal curve for Example 6.3



Areas to the left of $z = 0$ are found using negative values of z .

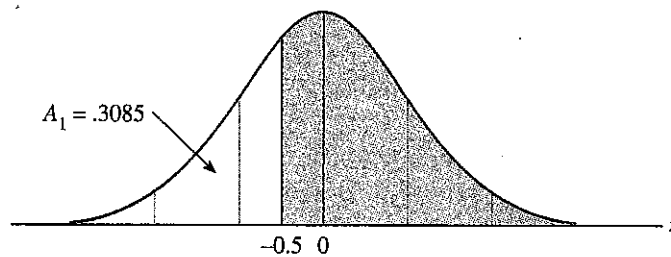
EXAMPLE

6.4

Find $P(z \geq -.5)$. This probability corresponds to the area to the *right* of a point $z = -.5$ standard deviation to the left of the mean (see Figure 6.10).

FIGURE 6.10

Area under the standard normal curve for Example 6.4



Solution The area given in Table 3 is the area to the left of a specified value of z . Indexing $z = -.5$ in Table 3, we can find the area A_1 to the *left* of $-.5$ to be .3085.

Since the area under the curve is 1, we find

$$P(z \geq -.5) = 1 - A_1 = 1 - .3085 = .6915.$$

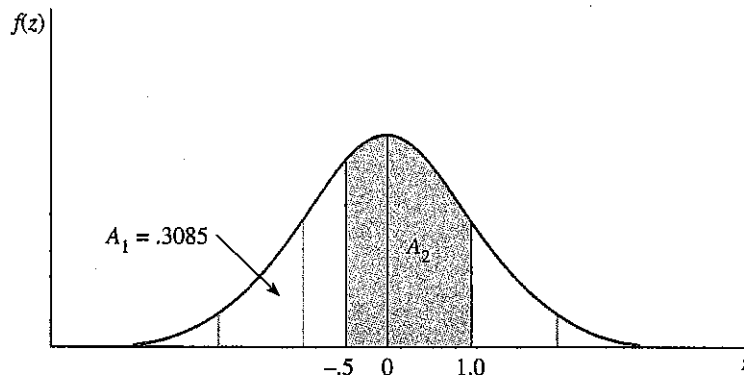
EXAMPLE

6.5

Find $P(-.5 \leq z \leq 1.0)$. This probability is the area between $z = -.5$ and $z = 1.0$, as shown in Figure 6.11.

FIGURE 6.11

Area under the standard normal curve for Example 6.5



Solution The area required is the shaded area A_2 in Figure 6.11. From Table 3 in Appendix I, you can find the area to the left of $z = -.5$ ($A_1 = .3085$) and the area to

the left of $z = 1.0$ ($A_1 + A_2 = .8413$). To find the area marked A_2 , we subtract the two entries:

$$A_2 = (A_1 + A_2) - A_1 = .8413 - .3085 = .5328$$

That is, $P(-.5 \leq z \leq 1.0) = .5328$.

MY

PERSONAL TRAINER

How Do I Use Table 3 to Calculate Probabilities under the Standard Normal Curve?

- To calculate the area to the left of a z -value, find the area directly from Table 3.
- To calculate the area to the right of a z -value, find the area in Table 3, and subtract from 1.
- To calculate the area between two values of z , find the two areas in Table 3, and subtract one area from the other.

Exercise Reps

Consider a standard random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$. Use Table 3 and fill in the probabilities below. The third probability is calculated for you.

The Interval	Write the Probability	Rewrite the Probability (if needed)	Find the Probability
Less than 1.5	$P(z < \underline{\quad})$		
Greater than 2	$P(z > \underline{\quad})$		
Greater than 2.33	$P(z > 2.33)$	$1 - P(z \leq 2.33)$	$1 - .9901 = .0099$
Between -1.96 and 1.96	$P(\underline{\quad} < z < \underline{\quad})$		
Between -1.24 and 2.37	$P(\underline{\quad} < z < \underline{\quad})$		
Less than or equal to -1	$P(z \leq \underline{\quad})$		

Progress Report

- Still having trouble? Try again using the Exercise Reps at the end of this section.
- Mastered the z -table? You can skip the Exercises Reps at the end of this section!

Answers are located on the perforated card at the back of this book.

EXAMPLE

6.6

Find the probability that a normally distributed random variable will fall within these ranges:

1. One standard deviation of its mean
2. Two standard deviations of its mean

Solution

1. Since the standard normal random variable z measures the distance from the mean in units of standard deviations, you need to find

$$P(-1 \leq z \leq 1) = .8413 - .1587 = .6826$$

Remember that you calculate the area between two z -values by subtracting the tabled entries for the two values.

2. As in part 1, $P(-2 \leq z \leq 2) = .9772 - .0228 = .9544$.

These probabilities agree with the approximate values of 68% and 95% in the Empirical Rule from Chapter 2.

EXAMPLE**6.7**

Find the value of z —say z_0 —such that .95 of the area is within $\pm z_0$ standard deviations of the mean.

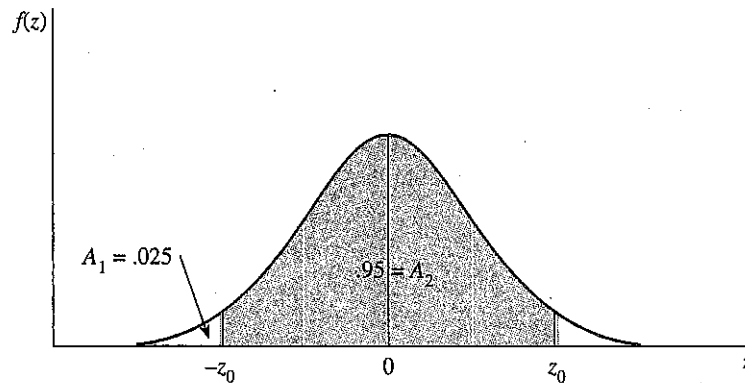


We know the area. Work from the inside of the table out.

Solution The shaded area in Figure 6.12 is the area within $\pm z_0$ standard deviations of the mean, which needs to be equal to .95. The “tail areas” under the curve are not shaded, and have a combined area of $1 - .95 = .05$. Because of the symmetry of the normal curve, these two tail areas have the same area, so that $A_1 = .05/2 = .025$ in Figure 6.12. Thus, the entire *cumulative area* to the left of z_0 to equal $A_1 + A_2 = .025 + .95 = .9750$. This area is found in the interior of Table 3 in Appendix I in the row corresponding to $z = 1.9$ and the .06 column. Hence, $z_0 = 1.96$. Note that this result is very close to the approximate value, $z = 2$, used in the Empirical Rule.

FIGURE 6.12

Area under the standard normal curve for Example 6.7



Calculating Probabilities for a General Normal Random Variable

Most of the time, the probabilities you are interested in will involve x , a normal random variable with mean μ and standard deviation σ . You must then *standardize* the interval of interest, writing it as the equivalent interval in terms of z , the standard normal random variable. Once this is done, the probability of interest is the area that you find using the *standard normal probability distribution*.

EXAMPLE**6.8**

Let x be a normally distributed random variable with a mean of 10 and a standard deviation of 2. Find the probability that x lies between 11 and 13.6.

Solution The interval from $x = 11$ to $x = 13.6$ must be standardized using the formula for z . When $x = 11$,

$$z = \frac{x - \mu}{\sigma} = \frac{11 - 10}{2} = .5$$

and when $x = 13.6$,

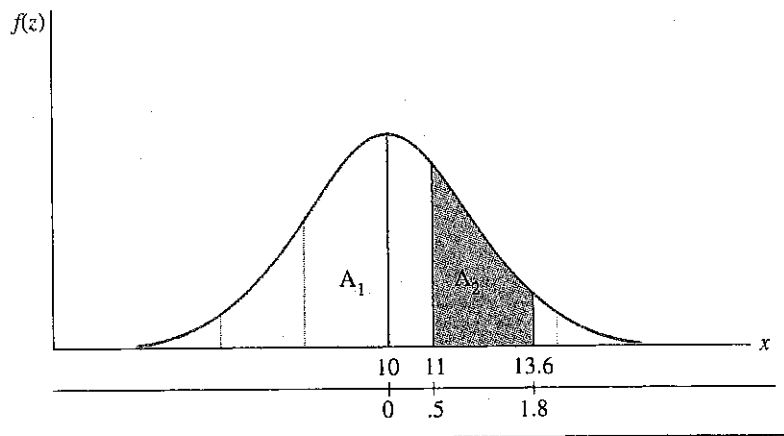
$$z = \frac{x - \mu}{\sigma} = \frac{13.6 - 10}{2} = 1.8$$

The desired probability is therefore $P(.5 \leq z \leq 1.8)$, the area lying between $z = .5$ and $z = 1.8$, as shown in Figure 6.13. From Table 3 in Appendix I, you find that the area to the left of $z = .5$ is .6915, and the area to the left of $z = 1.8$ is .9641. The desired probability is the difference between these two probabilities, or

$$P(.5 \leq z \leq 1.8) = .9641 - .6915 = .2726$$

FIGURE 6.13

Area under the standard normal curve for Example 6.8

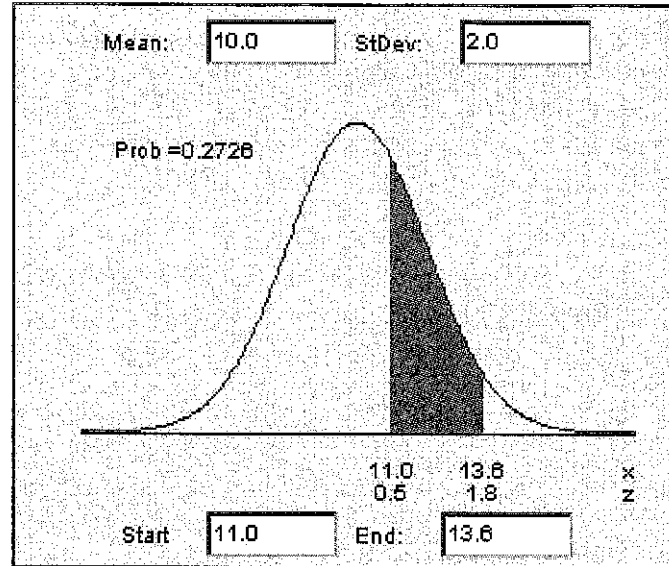


APPLET

The Java applet called **Normal Distribution Probabilities** allows you to calculate areas under a normal distribution for any values of μ and σ you select. Simply type the appropriate mean and standard deviation into the boxes at the top of the applet, type the interval of interest into the boxes at the bottom of the applet, and press “Enter” at each step to record your changes. (The “Tab” key will move your cursor from box to box.) The necessary area will be shaded in red on your monitor (light blue in Figure 6.14) and the probability is given to the left of the curve.

- If you need an area under the standard normal distribution, use $\mu = 0$ and $\sigma = 1$.
- In Example 6.8, we need an area under a normal distribution with $\mu = 10$ and $\sigma = 2$. Notice the values of x and z located along the horizontal axis. Find the probability, $P(11 \leq x \leq 13.6) = P(0.5 \leq z \leq 1.8) = .2726$, in Figure 6.14.

FIGURE 6.14
Normal Distribution
Probabilities applet



EXAMPLE 6.9

Studies show that gasoline use for compact cars sold in the United States is normally distributed, with a mean of 25.5 miles per gallon (mpg) and a standard deviation of 4.5 mpg. What percentage of compacts get 30 mpg or more?

Solution The proportion of compacts that get 30 mpg or more is given by the shaded area in Figure 6.15. To solve this problem, you must first find the z -value corresponding to $x = 30$. Substituting into the formula for z , you get

$$z = \frac{x - \mu}{\sigma} = \frac{30 - 25.5}{4.5} = 1.0$$

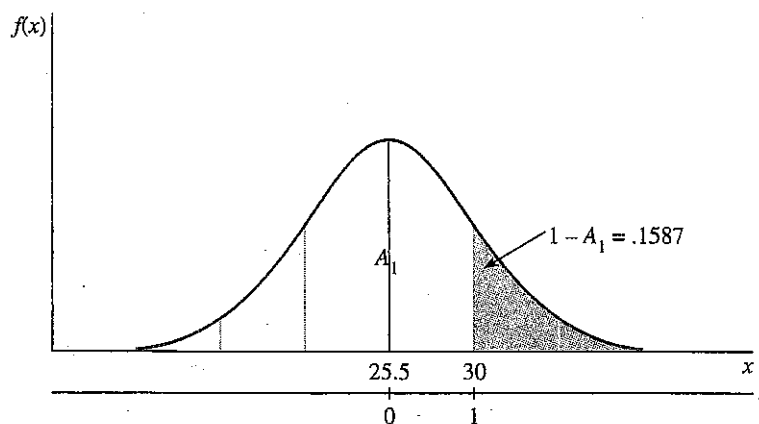
The area A_1 to the left of $z = 1.0$, is .8413 (from Table 3 in Appendix I). Then the proportion of compacts that get 30 mpg or more is equal to

$$P(x \geq 30) = 1 - P(z < 1) = 1 - .8413 = .1587$$

The percentage exceeding 30 mpg is

$$100(.1587) = 15.87\%$$

FIGURE 6.15
Area under the standard
normal curve for
Example 6.9



EXAMPLE

6.10

Refer to Example 6.9. In times of scarce energy resources, a competitive advantage is given to an automobile manufacturer who can produce a car that has substantially better fuel economy than the competitors' cars. If a manufacturer wishes to develop a compact car that outperforms 95% of the current compacts in fuel economy, what must the gasoline use rate for the new car be?

Solution The gasoline use rate x has a normal distribution with a mean of 25.5 mpg and a standard deviation of 4.5 mpg. You need to find a particular value—say, x_0 —such that

$$P(x \leq x_0) = .95$$

This is the 95th percentile of the distribution of gasoline use rate x . Since the only information you have about normal probabilities is in terms of the standard normal random variable z , start by standardizing the value of x_0 :

$$z_0 = \frac{x_0 - 25.5}{4.5}$$

Since the value of z_0 corresponds to x_0 , it must *also* have area .95 to its left, as shown in Figure 6.16. If you look in the interior of Table 3 in Appendix I, you will find that the area .9500 is exactly halfway between the areas for $z = 1.64$ and $z = 1.65$. Thus, z_0 must be exactly halfway between 1.64 and 1.65, or

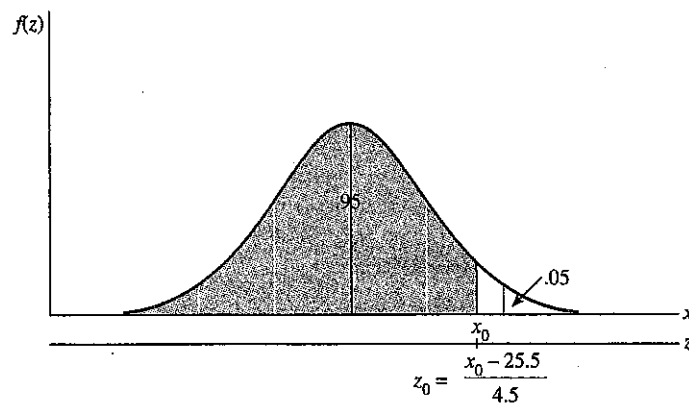
$$z_0 = \frac{x_0 - 25.5}{4.5} = 1.645$$

Solving for x_0 , you obtain

$$x_0 = \mu + z_0\sigma = 25.5 + (1.645)(4.5) = 32.9$$

FIGURE 6.16

Area under the standard normal curve for Example 6.10



The manufacturer's new compact car must therefore get 32.9 mpg to outperform 95% of the compact cars currently available on the U.S. market.

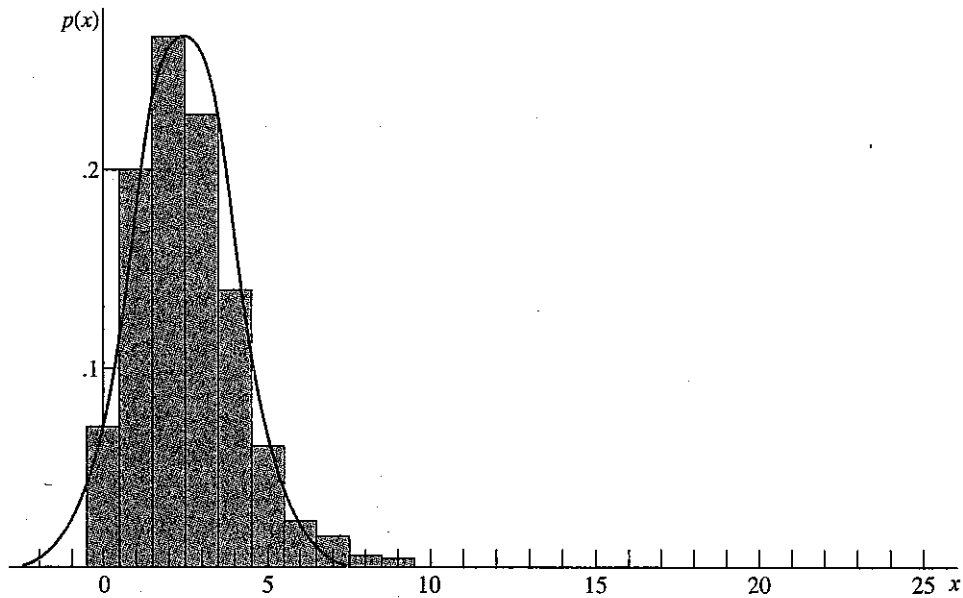
MY

APPLET

The Java applet called **Normal Probabilities and z-Scores** allows you to calculate areas under a normal distribution for any values of μ and σ you select. Once you specify one value for x , the applet calculates the value of z , and one of four types of areas, which you can select from the dropdown list at the bottom of the applet:

FIGURE 6.19

The binomial probability distribution and the approximating normal distribution for $n = 25$ and $p = .1$



If you superimpose a normal curve with the same mean, $\mu = np$, and the same standard deviation, $\sigma = \sqrt{npq}$, over the top of the bars, it “fits” quite well; that is, the areas under the curve are almost the same as the areas under the bars. However, when the probability of success, p , gets small and the distribution is skewed, as in Figure 6.19, the symmetric normal curve no longer fits very well. If you try to use the normal curve areas to approximate the area under the bars, your approximation will not be very good.

EXAMPLE

6.11

Use the normal curve to approximate the probability that $x = 8, 9$, or 10 for a binomial random variable with $n = 25$ and $p = .5$. Compare this approximation to the exact binomial probability.

Solution You can find the exact binomial probability for this example because there are cumulative binomial tables for $n = 25$. From Table 1 in Appendix I,

$$P(x = 8, 9, \text{ or } 10) = P(x \leq 10) - P(x \leq 7) = .212 - .022 = .190$$

To use the normal approximation, first find the appropriate mean and standard deviation for the normal curve:

$$\mu = np = 25(.5) = 12.5$$

$$\sigma = \sqrt{npq} = \sqrt{25(.5)(.5)} = 2.5$$

The probability that you need corresponds to the area of the three rectangles lying over $x = 8, 9$, and 10 . The equivalent area under the normal curve lies between $x = 7.5$ (the lower edge of the rectangle for $x = 8$) and $x = 10.5$ (the upper edge of the rectangle for $x = 10$). This area is shaded in Figure 6.18.

MY TIP

Only use the continuity correction if x has a binomial distribution!

To find the normal probability, follow the procedures of Section 6.3. First you standardize each interval endpoint:

$$z = \frac{x - \mu}{\sigma} = \frac{7.5 - 12.5}{2.5} = -2.0$$

$$z = \frac{x - \mu}{\sigma} = \frac{10.5 - 12.5}{2.5} = -.8$$

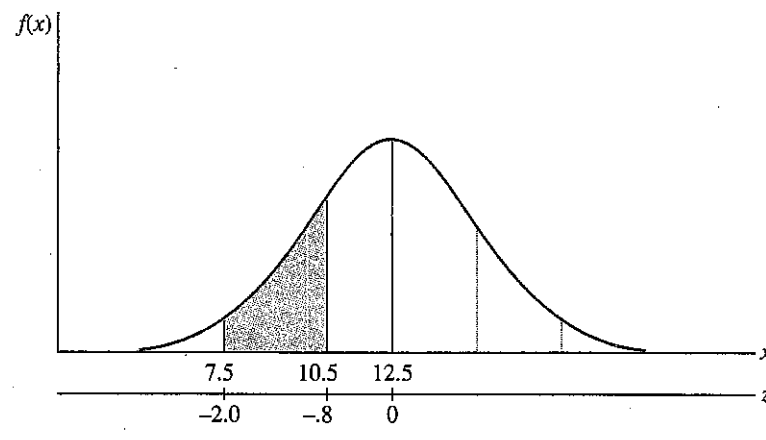
Then the approximate probability (shaded in Figure 6.20) is found from Table 3 in Appendix I:

$$P(-2.0 < z < -.8) = .2119 - .0228 = .1891$$

You can compare the approximation, .1891, to the actual probability, .190. They are quite close!

FIGURE 6.20

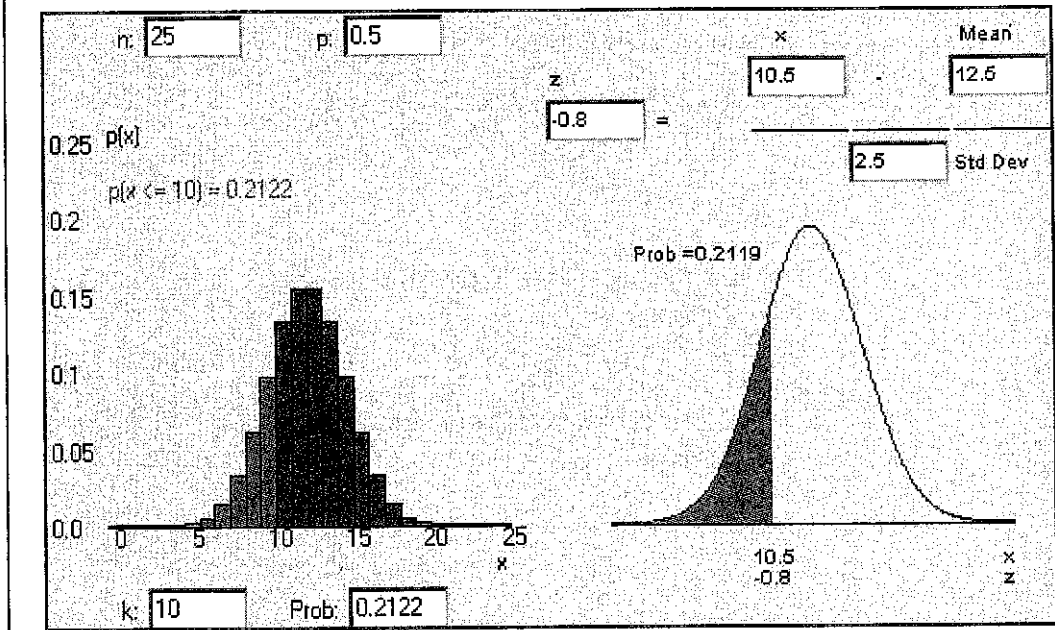
Area under the normal curve for Example 6.11



MY APPLET

You can use the Java applet called **Normal Approximation to Binomial Probabilities** shown in Figure 6.21 to compare the actual and approximate probabilities for the binomial distribution in Example 6.11. Enter the appropriate values of n and p in the boxes at the top left corner of the applet, and press “Enter” to record each entry. The exact binomial distribution on the left of the applet will change depending on the value of n you have entered. Now change the value of k in the box at the bottom left corner of the applet, and press “Enter.” The applet will calculate the exact binomial probability $P(x \leq k)$ in the box marked “Prob:” It will also calculate the approximate probability using the area under the normal curve. The z -value, with the continuity correction is shown at the top right, and the approximate probability is shown to the left of the normal curve. For Example 6.11, the applet calculates the normal approximation as $P(x \leq 10) \approx .2119$. What is the exact value of $P(x \leq 10)$? If you change k to 7 and press “Enter,” what is the approximate value for $P(x \leq 7)$? Now calculate $P(8 \leq x \leq 10)$. Does it match the answer we got in Example 6.11? You will use this applet again for the MyApplet Exercises section at the end of the chapter.

FIGURE 6.21
Normal Approximation
to Binomial Probabilities
applet



You must be careful not to exclude half of the two extreme probability rectangles when you use the normal approximation to the binomial probability distribution. This adjustment, called the **continuity correction**, helps account for the fact that you are approximating a *discrete random variable* with a *continuous* one. If you forget the correction, your approximation will not be very good! Use this correction only for *binomial probabilities*; do not try to use it when the random variable is already continuous, such as a height or weight.

How can you tell when it is appropriate to use the normal approximation to binomial probabilities? The normal approximation works well when the binomial histogram is roughly symmetric. This happens when the binomial distribution is not “bunched up” near 0 or n —that is, when it can spread out at least two standard deviations from its mean without exceeding its limits, 0 and n . Using this criterion, you can derive this simple rule of thumb:

RULE OF THUMB

The normal approximation to the binomial probabilities will be adequate if both

$$np > 5 \quad \text{and} \quad nq > 5$$

MY

PERSONAL TRAINER

How Do I Calculate Binomial Probabilities Using the Normal Approximation?

- Find the necessary values of n and p . Calculate $\mu = np$ and $\sigma = \sqrt{npq}$.
- Write the probability you need in terms of x and locate the appropriate area on the curve.
- Correct the value of x by ± 0.5 to include the entire block of probability for that value. This is the *continuity correction*.

- Convert the necessary x -values to z -values using

$$z = \frac{x \pm .5 - np}{\sqrt{npq}}$$

- Use Table 3 in Appendix I to calculate the approximate probability.

Exercise Reps

Consider a binomial random variable with $n = 30$ and $p = .4$. Fill in the blanks below to find some probabilities using the normal approximation.

A. Preliminary Steps:

1. Can we use the normal approximation? Calculate $np = \underline{\hspace{2cm}}$ and $nq = \underline{\hspace{2cm}}$
2. Are np and nq both greater than 5? Yes No
3. If the answer to Question 2 is yes, calculate $\mu = np = \underline{\hspace{2cm}}$ and $\sigma = \sqrt{npq} = \underline{\hspace{2cm}}$

B. Calculate the Probability:

1. To find the probability of 20 or more successes, what values of x should be included? $x = \underline{\hspace{2cm}}$
2. To include the entire block of probability for the first value of $x = \underline{\hspace{2cm}}$, start at $\underline{\hspace{2cm}}$.
3. Calculate $z = \frac{x \pm .5 - np}{\sqrt{npq}} = \underline{\hspace{2cm}}$
4. Calculate $P(x \geq 20) \approx P(z > \underline{\hspace{2cm}}) = 1 - \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$.

Progress Report

- Still having trouble? Try again using the Exercise Reps at the end of this section.
- Mastered the normal approximation? You can skip the Exercise Reps at the end of this section!

Answers are located on the perforated card at the back of this book.

EXAMPLE 6.12

The reliability of an electrical fuse is the probability that a fuse, chosen at random from production, will function under its designed conditions. A random sample of 1000 fuses was tested and $x = 27$ defectives were observed. Calculate the approximate probability of observing 27 or more defectives, assuming that the fuse reliability is .98.

Solution The probability of observing a defective when a single fuse is tested is $p = .02$, given that the fuse reliability is .98. Then

$$\begin{aligned}\mu &= np = 1000(.02) = 20 \\ \sigma &= \sqrt{npq} = \sqrt{1000(.02)(.98)} = 4.43\end{aligned}$$

The probability of 27 or more defective fuses, given $n = 1000$, is

$$P(x \geq 27) = p(27) + p(28) + p(29) + \cdots + p(999) + p(1000)$$



If np and nq are both greater than 5, you can use the normal approximation.

It is appropriate to use the normal approximation to the binomial probability because

$$np = 1000(.02) = 20 \quad \text{and} \quad nq = 1000(.98) = 980$$

are both greater than 5. The normal area used to approximate $P(x \geq 27)$ is the area under the normal curve to the right of 26.5, so that the entire rectangle for $x = 27$ is included. Then, the z -value corresponding to $x = 26.5$ is

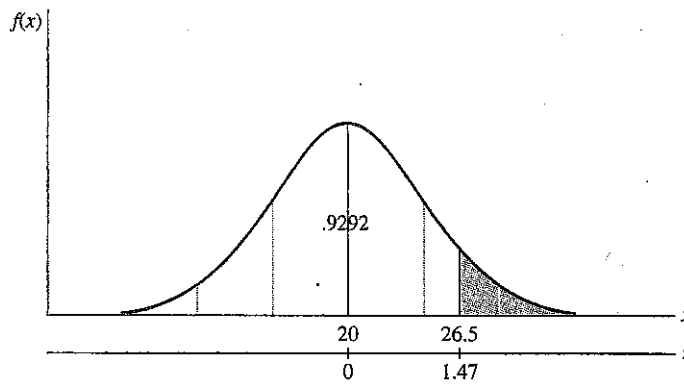
$$z = \frac{x - \mu}{\sigma} = \frac{26.5 - 20}{4.43} = \frac{6.5}{4.43} = 1.47$$

and the area to the left of $z = 1.47$ is equal to .9292, as shown in Figure 6.22. Since the total area under the curve is 1, you have

$$P(x \geq 27) \approx P(z \geq 1.47) = 1 - .9292 = .0708$$

FIGURE 6.22

Normal approximation to the binomial for Example 6.12



EXAMPLE

6.13

A producer of soft drinks was fairly certain that her brand had a 10% share of the soft drink market. In a market survey involving 2500 consumers of soft drinks, $x = 211$ expressed a preference for her brand. If the 10% figure is correct, find the probability of observing 211 or fewer consumers who prefer her brand of soft drink.

Solution If the producer is correct, then the probability that a consumer prefers her brand of soft drink is $p = .10$. Then

$$\mu = np = 2500(.10) = 250$$

$$\sigma = \sqrt{npq} = \sqrt{2500(.10)(.90)} = 15$$

The probability of observing 211 or fewer who prefer her brand is

$$P(x \leq 211) = p(0) + p(1) + \cdots + p(210) + p(211)$$

The normal approximation to this probability is the area to the left of 211.5 under a normal curve with a mean of 250 and a standard deviation of 15. First calculate

$$z = \frac{x - \mu}{\sigma} = \frac{211.5 - 250}{15} = -2.57$$

Then

$$P(x \leq 211) \approx P(z < -2.57) = .0051$$

The probability of observing a sample value of 211 or less when $p = .10$ is so small that you can conclude that one of two things has occurred: Either you have observed