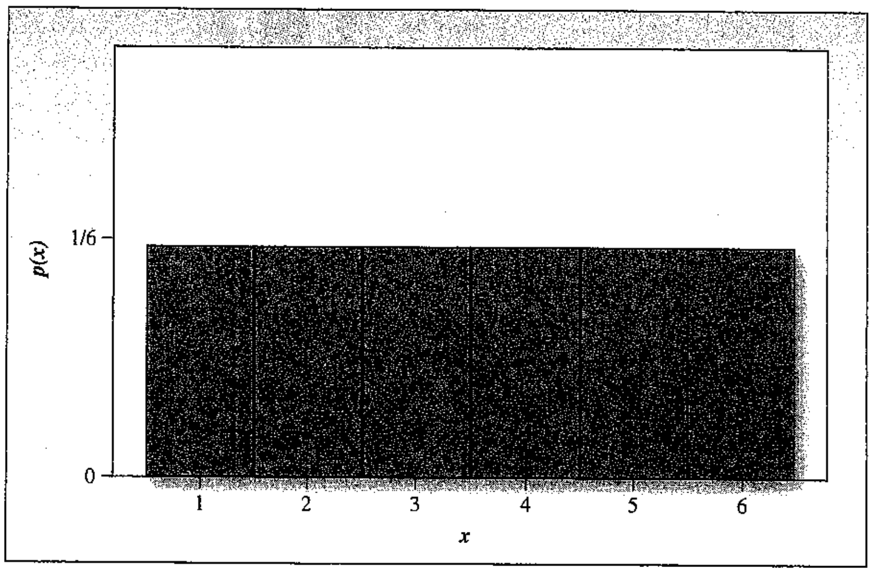
7.4 THE CENTRAL LIMIT THEOREM

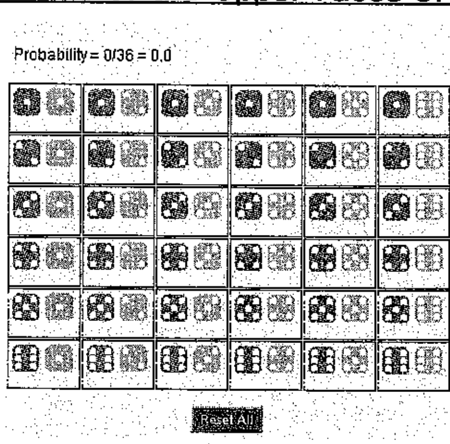
The **Central Limit Theorem** states that, under rather general conditions, sums and means of random samples of measurements drawn from a population tend to have an approximately normal distribution. Suppose you toss a balanced die $n = 1$ time. The random variable x is the number observed on the upper face. This familiar random variable can take six values, each with probability 1/6, and its probability distribution is shown in Figure 7.3. The shape of the distribution is *flat* or *uniform* and symmetric about the mean $\mu = 3.5$, with a standard deviation $\sigma = 1.71$. (See Section 4.8 and Exercise 4.84.)

FIGURE 7.3
Probability distribution for x, the number appearing on a single toss of a die



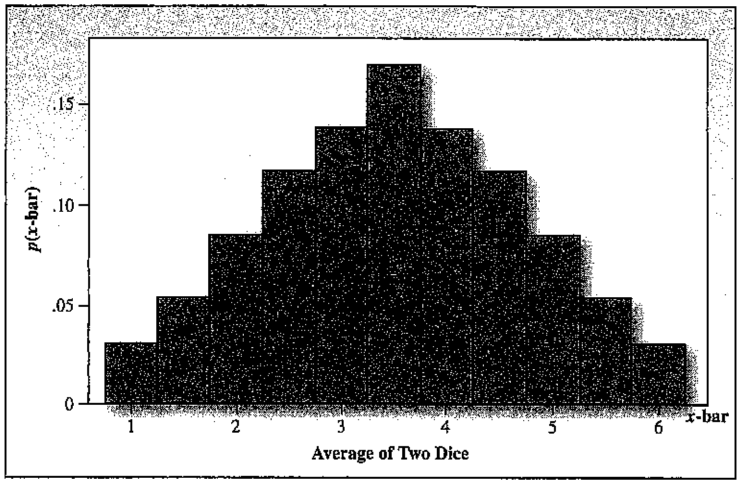
Now, take a sample of size $n = 2$ from this population; that is, toss two dice and record the sum of the numbers on the two upper faces, $\Sigma x_i = x_1 + x_2$. Table 7.5 shows the 36 possible outcomes, each with probability 1/36. The sums are tabulated, and each of the possible sums is divided by $n = 2$ to obtain an average. The result is the **sampling distribution** of $\bar{x} = \Sigma x_i/n$, shown in Figure 7.4. You should notice the dramatic difference in the shape of the sampling distribution. It is now roughly mound-shaped but still symmetric about the mean $\mu = 3.5$.

TABLE 7.5 **Sums of the Upper Faces of Two Dice**



	First Die					
Second Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

FIGURE 7.4
Sampling distribution of $\bar{x}$ for $n = 2$ dice



Using *MINITAB,* we generated the sampling distributions of $\bar{x}$ when $n = 3$ and $n = 4$. For $n = 3$, the sampling distribution in Figure 7.5 clearly shows the mound shape of the normal probability distribution, still centered at $\mu = 3.5$. Notice also that the spread of the distribution is slowly *decreasing* as the sample size *n increases.* Figure 7.6 dramatically shows that the distribution of $\bar{x}$ is approximately normally distributed based on a sample as small as $n = 4$. This phenomenon is the result of an important statistical theorem called the **Central Limit Theorem (CLT).**

FIGURE 7.5
MINITAB sampling distribution of $\bar{x}$ for $n = 3$ dice

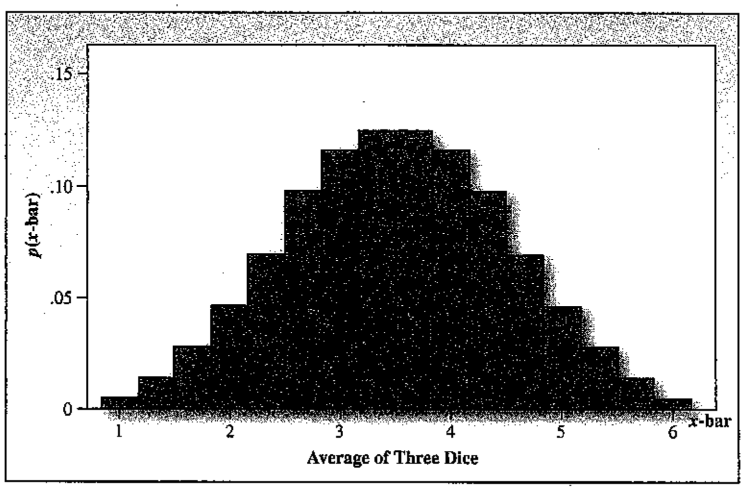
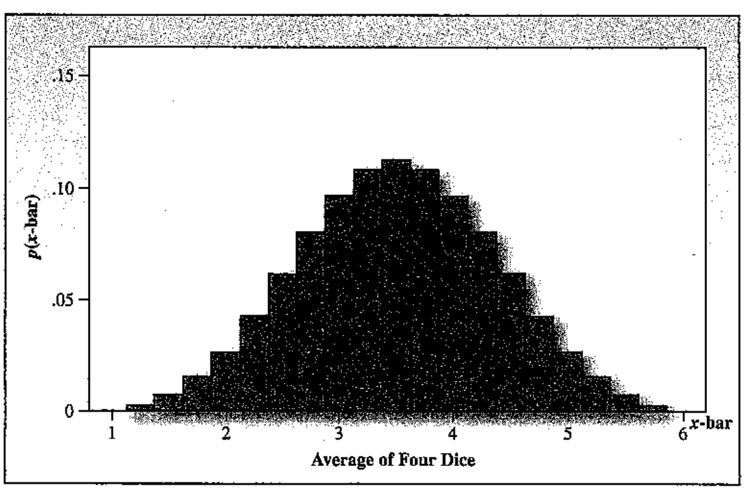


FIGURE 7.6
MINITAB sampling distribution of $\bar{x}$ for $n = 4$ dice



Central Limit Theorem

If random samples of n observations are drawn from a nonnormal population with finite mean μ and standard deviation σ, then, when n is large, the sampling distribution of the sample mean $\bar{x}$ is approximately normally distributed, with mean μ and standard deviation

$$\frac{\sigma}{\sqrt{n}}$$

The approximation becomes more accurate as n becomes large.

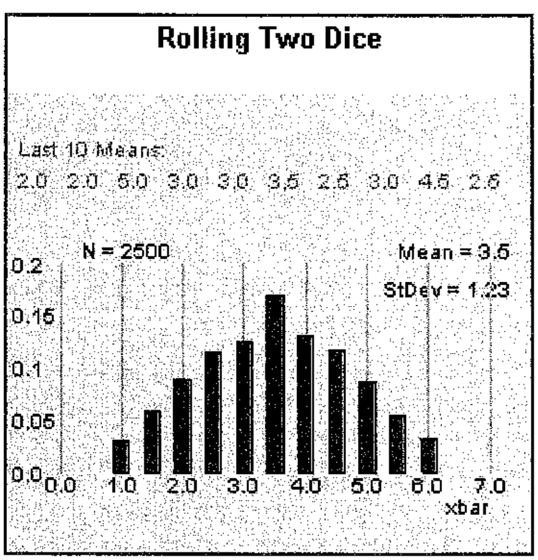
Regardless of its shape, the sampling distribution of $\bar{x}$ always has a mean identical to the mean of the sampled population and a standard deviation equal to the population standard deviation σ divided by $\sqrt{n}$. Consequently, *the spread of the distribution of sample means is considerably less than the spread of the sampled population.*

The Central Limit Theorem can be restated to apply to the **sum of the sample measurements** Σx_i, which, as n becomes large, also has an approximately normal distribution with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$.

MY APPLET

The Java applet called **The Central Limit Theorem** can be used to perform a *simulation* for the sampling distributions of the average of one, two, three or four dice. Figure 7.7 shows the applet after the pair of dice ($n = 2$) has been tossed 2500 times. This is not as hard as it seems, since you need only press the Roll 100 Sets button 25 times. The simulation shows the possible values for $\bar{x} = \Sigma x_i/10$ and also shows the mean and standard deviation for these 2500 measurements. The mean, 3.5, is exactly equal to $\mu = 3.5$. What is the standard deviation for these 2500 measurements? Is it close to the theoretical value, $\sigma/\sqrt{n} = \frac{1.71}{\sqrt{2}} = 1.21$? You will use this applet again for the MyApplet Exercises at the end of the chapter.

FIGURE 7.7
Central Limit Theorem applet



TIP
The sampling distribution of $\bar{x}$ always has a mean μ and standard deviation $\sigma/\sqrt{n}$. The CLT helps describe its shape.

The important contribution of the Central Limit Theorem is in statistical inference. Many estimators that are used to make inferences about population parameters are sums or averages of the sample measurements. When the sample size is sufficiently large, you can expect these estimators to have sampling distributions that are approximately normal. You can then use the normal distribution to describe the behavior of these estimators in repeated sampling and evaluate the probability of observing certain sample results. As in Chapter 6, these probabilities are calculated using the standard normal random variable

$$z = \frac{\text{Estimator} - \text{Mean}}{\text{Standard deviation}}$$

As you reread the Central Limit Theorem, you may notice that the approximation is valid as long as the sample size n is "large"—but how large is "large"? Unfortunately, there is no clear answer to this question. The appropriate value of n depends on the shape of the population from which you sample as well as on how you want to use the approximation. However, these guidelines will help:

HOW DO I DECIDE WHEN THE SAMPLE SIZE IS LARGE ENOUGH?

- If the sampled population is **normal,** then the sampling distribution of $\bar{x}$ will also be normal, no matter what sample size you choose. This result can be proven theoretically, but it should not be too difficult for you to accept without proof.
- When the sampled population is approximately **symmetric,** the sampling distribution of $\bar{x}$ becomes approximately normal for relatively small values of n. Remember how rapidly the "flat" distribution in the dice example became mound-shaped ($n = 3$).
- When the sampled population is **skewed,** the sample size n must be larger, with n at least 30 before the sampling distribution of $\bar{x}$ becomes approximately normal.

These guidelines suggest that, for many populations, the sampling distribution of $\bar{x}$ will be approximately normal for moderate sample sizes; an exception to this rule occurs in sampling a binomial population when either p or $q = (1 - p)$ is very small. As specific applications of the Central Limit Theorem arise, we will give you the appropriate sample size n.

7.5 THE SAMPLING DISTRIBUTION OF THE SAMPLE MEAN

If the population mean μ is unknown, you might choose several *statistics* as an estimator; the sample mean $\bar{x}$ and the sample median m are two that readily come to mind. Which should you use? Consider these criteria in choosing the estimator for μ:

- Is it easy or hard to calculate?
- Does it produce estimates that are consistently too high or too low?
- Is it more or less variable than other possible estimators?

The sampling distributions for $\bar{x}$ and m with $n = 3$ for the small population in Example 7.3 showed that, in terms of these criteria, the sample mean performed better than the sample median as an estimator of μ. In many situations, the sample mean $\bar{x}$ has desirable properties as an estimator that are not shared by other competing estimators; therefore, it is more widely used.

THE SAMPLING DISTRIBUTION OF THE SAMPLE MEAN, $\bar{x}$

- If a random sample of n measurements is selected from a population with mean μ and standard deviation σ, the sampling distribution of the sample mean $\bar{x}$ will have mean μ and standard deviation†

$$\frac{\sigma}{\sqrt{n}}$$

- If the population has a *normal* distribution, the sampling distribution of $\bar{x}$ will be *exactly* normally distributed, *regardless of the sample size, n.*
- If the population distribution is *nonnormal,* the sampling distribution of $\bar{x}$ will be *approximately* normally distributed for large samples (by the Central Limit Theorem).

Standard Error

Definition The standard deviation of a statistic used as an estimator of a population parameter is also called the **standard error of the estimator** (abbreviated **SE**) because it refers to the precision of the estimator. Therefore, the standard deviation of $\bar{x}$—given by $\sigma/\sqrt{n}$—is referred to as the **standard error of the mean** (abbreviated as SE($\bar{x}$) or just SE).

†When repeated samples of size n are randomly selected from a *finite* population with N elements whose mean is μ and whose variance is σ^2, the standard deviation of $\bar{x}$ is

$$\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

where σ^2 is the population variance. When N is large relative to the sample size n, $\sqrt{(N-n)(N-1)}$ is approximately equal to 1, and the standard deviation of $\bar{x}$ is

$$\frac{\sigma}{\sqrt{n}}$$

a. What is the probability that a randomly selected sample of 31 patients would produce an average diameter of 6.5 mm or less for the nonaffected tendon?

b. When the diameters of the affected tendon were measured for a sample of 31 patients, the average diameter was 9.80. If the average tendon diameter in the population of patients with AT is no different than the average diameter of the nonaffected tendons (5.97 mm), what is the probability of observing an average diameter of 9.80 or higher?

c. What conclusions might you draw from the results of part b?

7.6 THE SAMPLING DISTRIBUTION OF THE SAMPLE PROPORTION

There are many practical examples of the binomial random variable x. One common application involves consumer preference or opinion polls, in which we use a random sample of n people to estimate the proportion p of people in the population who have a specified characteristic. If x of the sampled people have this characteristic, then the sample proportion

$$\hat{p} = \frac{x}{n}$$

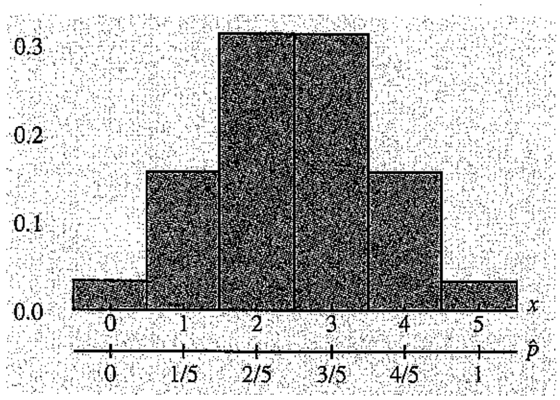
can be used to estimate the population proportion p (Figure 7.12).†

MY TIP

Q: How do you know if it's binomial or not?
A: Look to see if the measurement taken on a single experimental unit in the sample is a "success/failure" type. If so, it's probably binomial.

The binomial random variable x has a probability distribution $p(x)$, described in Chapter 5, with mean np and standard deviation $\sqrt{npq}$. Since $\hat{p}$ is simply the value of x, expressed as a proportion $\left(\hat{p} = \frac{x}{n}\right)$, the sampling distribution of $\hat{p}$ is identical to the probability distribution of x, except that it has a new scale along the horizontal axis.

FIGURE 7.12
Sampling distribution of the binomial random variable x and the sample proportion $\hat{p}$



Because of this change of scale, the mean and standard deviation of $\hat{p}$ are also rescaled, so that the mean of the sampling distribution of $\hat{p}$ is p, and its standard error is

$$SE(\hat{p}) = \sqrt{\frac{pq}{n}} \qquad \text{where } q = 1 - p$$

Finally, just as we can approximate the probability distribution of x with a normal distribution when the sample size n is large, we can do the same with the sampling distribution of $\hat{p}$.

†A "hat" placed over the symbol of a population parameter denotes a statistic used to estimate the population parameter. For example, the symbol $\hat{p}$ denotes the sample proportion.

PROPERTIES OF THE SAMPLING DISTRIBUTION OF THE SAMPLE PROPORTION, $\hat{p}$

- If a random sample of n observations is selected from a binomial population with parameter p, then the sampling distribution of the sample proportion

  $$\hat{p} = \frac{x}{n}$$

  will have a mean

  $$p$$

  and a standard deviation

  $$\text{SE}(\hat{p}) = \sqrt{\frac{pq}{n}} \qquad \text{where } q = 1 - p$$

- When the sample size n is large, the sampling distribution of $\hat{p}$ can be approximated by a normal distribution. The approximation will be adequate if $np > 5$ and $nq > 5$.

EXAMPLE 7.6

In a survey, 500 mothers and fathers were asked about the importance of sports for boys and girls. Of the parents interviewed, 60% agreed that the genders are equal and should have equal opportunities to participate in sports. Describe the sampling distribution of the sample proportion $\hat{p}$ of parents who agree that the genders are equal and should have equal opportunities.

Solution You can assume that the 500 parents represent a random sample of the parents of all boys and girls in the United States and that the true proportion in the population is equal to some unknown value that you can call p. The sampling distribution of $\hat{p}$ can be approximated by a normal distribution,† with mean equal to p (see Figure 7.13) and standard error

$$\text{SE}(\hat{p}) = \sqrt{\frac{pq}{n}}$$

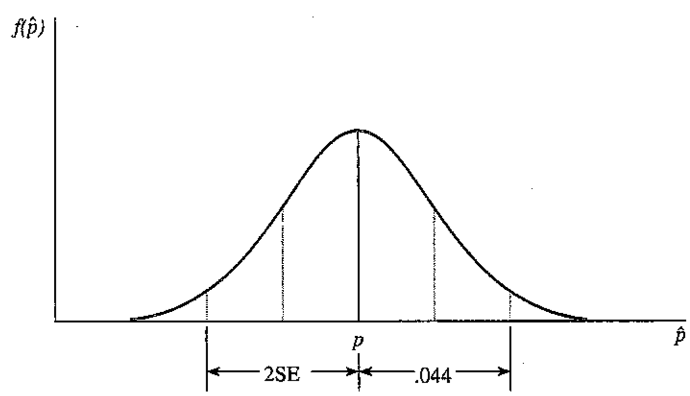


FIGURE 7.13 The sampling distribution for $\hat{p}$ based on a sample of $n = 500$ parents for Example 7.6

†Checking the conditions that allow the normal approximation to the distribution of $\hat{p}$, you can see that $n = 500$ is adequate for values of p near .60 because $n\hat{p} = 300$ and $n\hat{q} = 200$ are both greater than 5.

You can see from Figure 7.13 that the sampling distribution of $\hat{p}$ is centered over its mean p. Even though you do not know the exact value of p (the sample proportion $\hat{p} = .60$ may be larger or smaller than p), an approximate value for the standard deviation of the sampling distribution can be found using the sample proportion $\hat{p} = .60$ to approximate the unknown value of p. Thus,

$$\text{SE} = \sqrt{\frac{pq}{n}} \approx \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

$$= \sqrt{\frac{(.60)(.40)}{500}} = .022$$

Therefore, approximately 95% of the time, $\hat{p}$ will fall within 2SE $\approx$ .044 of the (unknown) value of p.

MY PERSONAL TRAINER

How Do I Calculate Probabilities for the Sample Proportion $\hat{p}$?

1. Find the necessary values of n and p.
2. Check whether the normal approximation to the binomial distribution is appropriate ($np > 5$ and $nq > 5$).
3. Write down the event of interest in terms of $\hat{p}$, and locate the appropriate area on the normal curve.
4. Convert the necessary values of $\hat{p}$ to z-values using

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

5. Use Table 3 in Appendix I to calculate the probability.

Exercise Reps (Fill in the Blanks)

A. You take a random sample of size $n = 36$ from a binomial distribution with mean $p = .4$. The sampling distribution of $\hat{p}$ will be approximately ________ with a mean of ____ and a standard deviation (or standard error) of ____.

B. To find the probability that the sample proportion exceeds .5, write down the event of interest. ____
When $\hat{p} = .5$,

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} = \_\_\_\_$$

Find the probability:

$$P(\hat{p} > \_\_\_\_) = P(z > \_\_\_\_) = 1 - \_\_\_\_ = \_\_\_\_$$

(continued)